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Confidence Intervals For P(X<Y) In The Exponential Case With Common Location Parameter

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The problem considered is interval estimation of the stress - strength reliability R = P(X<Y) where X and Y have independent exponential distributions with parameters θ and λ respectively and a common location parameter μ. Several types of asymptotic, approximate and bootstrap intervals are investigated. Performances are investigated using simulation techniques and compared in terms of attainment of the nominal confidence level, symmetry of lower and upper error rates, and expected length. Recommendations concerning their usage are given.

Key words: Bootstrap, exponential distribution, interval estimation, stress-strength model

Introduction

The problem of making inference about R = P(X<Y) has received a considerable attention in literature. This problem arises naturally in the context of mechanical reliability of a system with strength X and stress Y. The system fails any time its strength is exceeded by the stress applied to it. Another interpretation of R is that it measures the effect of the treatment when X is the response for a control group and Y refers to the treatment group. Beg (1980) obtained the (MVUE) of R when X and Y are independent exponential random variables with unequal scale and unequal location parameters.

Gupta and Gupta (1988) obtained the maximum likelihood estimator (MLE), the MVUE, and a Bayes estimator of R in case of different location parameters and a common scale parameter. Various other versions of this problem have been discussed in literature, see Johnson et al. (1994).

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The problem of developing confidence intervals for the stress - strength probability has received relatively little attention; Halperin (1987) and Hamdy (1995) developed distribution free confidence intervals, while Bai and Hong (1992) discussed point and interval estimation of in the case of two independent exponentials with common location parameter, they derived two types of approximate intervals but did not study their finite sample properties and did not give an idea about how do they compare with each other.

In this article, for the same problem considered by Bai and Hong (1992), we shall investigate and compare the performance of the two intervals of Bai and Hong together with some other types of confidence intervals like intervals based on the transformed maximum likelihood estimator, the likelihood ratio statistic and intervals based on the bootstrap (Efron & Tibshirani, 1993). The model and maximum likelihood estimation of its parameters will be presented in section 2. The “non-bootstrap” confidence intervals will be presented in section 3, while bootstrap intervals are discussed in section 4. A Monte Carlo study designed to investigate and compare the intervals is described in section 5. Results and conclusions are given in the final section.

The Model and Maximum Likelihood Estimation

In this study, X and Y are independently exponentially distributed random variables with scale parameters θ and λ respectively and a common location parameter μ, that is

$$f_X(x,\theta,\mu)=\theta e^{-\theta(x-\mu)},\ x\ge\mu\ ;$$
$$f_Y(y,\lambda,\mu)=\lambda e^{-\lambda(y-\mu)},\ y\ge\mu\ .$$

Let $X_1,...,X_{n_1}$ be a random sample for X and $Y_1,...,Y_{n_2}$ be a random sample for Y. The parameter R we want to estimate is $R=p(X<Y)=\dfrac{\theta}{\theta+\lambda}$. The likelihood function can be written as

$$L(\theta,\lambda,\mu)=\theta^{n_1}\lambda^{n_2}\exp\left(-\theta\sum_{i=1}^{n_1}(x_i-\mu)-\lambda\sum_{i=1}^{n_1}(y_i-\mu)\right)I(z\ge\mu)$$

where $z=\min(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2})$ and $I(.)$ indicates the usual indicator function.

The maximum likelihood estimators of $\theta,\lambda,$ and μ are given by (Ghosh & Razmpour, 1984) $\hat{\mu}=z$, $\hat{\theta}=\dfrac{n_1}{T_1}$, and $\hat{\lambda}=\dfrac{n_2}{T_2}$, where $T_1=\sum_{i=1}^{n_1}(x_i-z)$ and $T_2=\sum_{i=1}^{n_2}(y_i-z)$. The maximum likelihood estimator of R is therefore $\hat{R}=\dfrac{n_1T_2}{n_2T_1+n_1T_2}$. Now we will describe the various intervals under study.

Confidence Intervals for *R*

Exact confidence intervals that are convenient to use for *R* are not available and hence approximate methods that exist in a simple closed form are needed. In this section and the following section we shall develop various types of intervals for the stress – strength reliability (*R*).

Intervals Based on the Asymptotic Normality of the MLE (AN Intervals)

Bai and Hong (1992) showed that if $n=n_1+n_2\to\infty$ such that $\dfrac{n_1}{n}\to\gamma,\ 0<\gamma<1$. Then $\sqrt{n}(\hat{R}-R)\to N(0,\sigma^2)$ where $\sigma^2=\dfrac{R^2(1-R)^2}{\gamma(1-\gamma)}$. This fact can be used to construct approximate confidence intervals for R. The intervals are of the form

$$\left(\hat{R}\pm z_{1-\alpha/2}\frac{\hat{R}(1-\hat{R})}{\sqrt{n(n_1/n)(1-n_1/n)}}\right),$$

where $z_{1-\alpha/2}$ is the $1-\alpha/2$-quantile of the standard normal distribution.

Intervals Based on the Asymptotic Normality of the Transformed MLE (TRAN Intervals)

When the maximum likelihood estimator of the parameter of interest has its range in only a part of the real line, a monotone transformation of this parameter with continuous derivatives and range in the entire real line will generally be better approximated by an asymptotic normal distribution as suggested by many authors including Lawless (1982) and Nelson (1982). Let $K(R)$ be a monotone function of R and let $K'(R)$ be the first derivative, then by applying the delta method (Serfling, 1980) we get

$$\sqrt{n}\left(K(\hat{R})-K(R)\right)\to N\left(0,K'(R)^2V(\hat{R})\right).$$

Using this, a $1-\alpha$ confidence interval for *R* may be obtained as

$$\begin{pmatrix} K^{-1}\left(K(\hat{R})-z_{1-\alpha/2}gK(\hat{R})^2\hat{V}(\hat{R})\right), \\ K^{-1}\left(K(\hat{R})+zK'(\hat{R})^2\hat{V}(\hat{R})\right) \end{pmatrix}.$$

An appropriate transform is the $\tan^{-1}$ (Jeng & Meeker, 2003). Using this transform a $1-\alpha$ confidence interval for *R* is given by

$$\left\{ \tan\left(\frac{\tan^{-1}\left(\hat{R}\right) \pm z_{1-\alpha/2}}{\frac{\hat{R}\left(1-\hat{R}\right)}{\left(1+\hat{R}^2\right)\left(n\left(n_1/n\right)\left(1-n_1/n\right)\right)^{1/2}}} \right) \right\}.$$

Bai and Hong's Intervals (BH intervals)

Ghosh and Razmpour (1984) showed that $\left(T_1, T_2, Z\right)$ is a complete sufficient for (θ, λ, μ) and that the joint probability density function of $\left(T_1, T_2\right)$ which is independent of Z is

$$g(t_1, t_2) = \left(\frac{\theta^{n_1}\lambda^{n_2}}{n_1\theta + n_2\lambda}\right)\left(\frac{n_2 t_1^{n_1-1} t_2^{n_2-2}}{\Gamma(n_1)\Gamma(n_2-1)} + \frac{n_1 t_1^{n_1-2} t_2^{n_2-1}}{\Gamma(n_1-1)\Gamma(n_2)}\right), \exp(-\theta t_1 - \lambda t_2)$$

$$t_1, t_2 > 0.$$

Using standard transformation techniques, it can be shown that the probability density function of the random variable $U = \frac{\theta T_1}{\theta T_1 + \lambda T_2}$ is given by (Bai and Hong, 1992)

$$g(u, \pi, n_1, n_2) = \pi b(u, n_1 - 1, n_2) + (1-\pi) b(u, n_1, n_2 - 1), \quad 0 \le u \le 1.$$

where $\pi = \frac{n_1\theta}{n_1\theta + n_2\lambda}$ and

$$b(u, r, s) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} u^{r-1}(1-u)^{s-1}, \quad 0 \le u \le 1.$$

is the beta probability density function with parameters r and s. Bai and Hong (1992) showed that an approximate $1-\alpha$ interval for R is of the form

$$\left(\frac{k_{\alpha/2} t_2}{\left(1-k_{\alpha/2}\right)t_1 + k_{\alpha/2} t_2}, \frac{k_{1-\alpha/2} t_2}{\left(1-k_{1-\alpha/2}\right)t_1 + k_{1-\alpha/2} t_2}\right)$$

where t_1 and t_2 are the observed values of T_1 and T_2 respectively, and k_α is such that $G\left(k_\alpha, \hat{\pi}, n_1, n_2\right) = \alpha$. Here $\hat{\pi}$ is an estimator of π obtained by substituting the maximum likelihood estimators of θ and λ in the formula of π, and G is the distribution function of mixed beta random variable U.

Intervals Based on the Likelihood Ratio Statistic (LR Intervals)

The likelihood function of (θ, λ, μ) is given by

$$L(\theta, \lambda, \mu) = \theta^{n_1}\lambda^{n_2} \exp\left(-\theta\sum_{i=1}^{n_1}(x_i - \mu) - \lambda\sum_{i=1}^{n_1}(y_i - \mu)\right) I(z \ge \mu).$$

The likelihood ratio statistic for testing $H_0 : R = R_0$ is defined as (Barndorff-Nielsen and Cox, 1994) $W = 2\left(l(\Omega) - l(\varpi)\right)$, where $l(\Omega)$ is the log-likelihood function evaluated at the values of the unrestricted maximum likelihood estimator of (θ, λ, μ). While $l(\varpi)$ is the log-likelihood function evaluated at the values of the restricted maximum likelihood estimator under the null hypothesis. Recall that the unrestricted maximum likelihood estimators are $\hat{\mu} = z$, $\hat{\theta} = \frac{n_1}{T_1}$, and $\hat{\lambda} = \frac{n_2}{T_2}$, where $T_1 = \sum_{i=1}^{n_1}(x_i - z)$ and $T_2 = \sum_{i=1}^{n_2}(y_i - z)$. Under the null hypothesis $H_0 : R = R_0$ we find readily that $\lambda = \frac{1-R_0}{R_0}\theta$ and thus the maximum likelihood estimator of θ is $\tilde{\theta} = \frac{n_1 + n_2}{T_1 + \frac{1-R_0}{R_0}T_2}$. Substituting in the formula of the likelihood ratio statistic and simplifying we get

$$W(R_0)=2\left[\begin{array}{l} n_1\ln\left(\frac{n_1}{T_1}\right)+n_2\ln\left(\frac{n_2}{T_2}\right)- \\ \left(n_2\ln\left(\frac{1-R_0}{R_0}\right)+(n_1+n_2)\right. \\ \left.\ln\left(\frac{n_1+n_2}{T_1+\frac{1-R_0}{R_0}T_2}\right)\right) \end{array}\right].$$

The distribution of $W(R_0)$ is χ_1^2 (Barndorff-Nielsen and Cox, 1994). The bounds of likelihood ratio confidence intervals with $(1-\alpha)$nominal coverage probability are the two roots of $W(R_0)=\chi^2_{\alpha,1}$, where $\chi^2_{\alpha,1}$ is the upper α quantile of the chi square distribution with one degree of freedom.

Parametric Bootstrap Intervals

The following methods of deriving confidence intervals are based on the Bootstrap approach (Efron & Tibshirani, 1993). They are computer intensive methods based on resampling with replacement from the original data and then using these Bootstrap samples to study the behaviour of estimators and tests. When the parametric form of the distribution from which the data are generated is known except for some unknown parameters, we generate from this distribution after its parameters are replaced by their estimates. The advantage of bootstrap methods is their wide applicability and remarkable accuracy, especially in situations where the traditional methods do not work. There are several Bootstrap based intervals discussed in the literature (Efron and Tibshirani, 1993), the most common ones are the bootstrap –t interval, the percentile interval and the bias corrected and accelerated (BC_a) interval.

The Bootstrap – t Interval Based on the MLE (BTST Intervals)

Let $\hat{R}$ be the maximum likelihood estimator of R and let $\hat{R}^*$ be the maximum likelihood estimator calculated from the bootstrap sample. Let z^*_α be the α quantile of the bootstrap distribution of

$$Z^*=\frac{\left(\hat{R}^*-\hat{R}\right)}{\left(\hat{V}\left(\hat{R}^*\right)\right)^{\frac{1}{2}}},$$

where $\hat{V}\left(\hat{R}^*\right)$ is estimated variance of $\hat{R}$ calculated from the bootstrap sample. The bootstrap-t interval is given by $\left(\hat{R}-z^*_{\alpha/2}\hat{V}\left(\hat{R}\right),\hat{R}+z^*_{1-\alpha/2}\hat{V}\left(\hat{R}\right)\right)$ where z^*_α is determined by simulation.

The Bootstrap – t Interval Based on the Transformed MLE (TRBTST Intervals)

Let $\hat{R}$ be the maximum likelihood estimator of R and let $\hat{R}^*$ be the maximum likelihood estimator calculated from the bootstrap sample. Let z^*_α be the α quantile of the bootstrap distribution of

$$Q^*=\frac{\left(\tan^{-1}\left(\hat{R}^*\right)-\tan\left(\hat{R}\right)\right)}{\left(1+\hat{R}^{*2}\right)^{-1}\left(\hat{V}\left(\hat{R}^*\right)\right)^{\frac{1}{2}}},$$

where $\hat{V}\left(\hat{R}^*\right)$ is estimated variance of $\hat{R}$ calculated from the bootstrap sample. The bootstrap-t interval is given by

$$\left(\begin{array}{l}\hat{R}-q^*_{\alpha/2}V\left(\hat{R}\right)^{\frac{1}{2}}\left(1+\hat{R}^2\right)^{-1}, \\ \hat{R}+q^*_{1-\alpha/2}V\left(\hat{R}\right)^{\frac{1}{2}}\left(1+\hat{R}^2\right)^{-1}\end{array}\right)$$

where $q^*_{\alpha/2}$ and $q^*_{1-\alpha/2}$ are the quantiles of the bootstrap distribution of Q^* determined by simulation.

The Percentile Interval (PRC Interval)

Here we simulate the bootstrap distribution of $\hat{R}^*$ by resampling repeatedly from the parametric model of the original data and calculating $\hat{R}_i^*, i = 1, \ldots, B$ where B is the number of bootstrap samples. Let $\hat{H}$ be the cumulative distribution function of $\hat{R}^*$, then the $1-\alpha$ interval is given by $\left(\hat{H}^{-1}\left(\frac{\alpha}{2}\right),\ \hat{H}^{-1}\left(1-\frac{\alpha}{2}\right)\right)$.

The Bias Corrected and Accelerated Interval (**BCa** Interval)

The bias corrected and accelerated interval is calculated also using the percentiles of the bootstrap distribution of $\hat{R}^*$, but not necessarily identical with the percentile interval described in the previous subsection. The percentiles depend on two numbers $\hat{a}$ and $\hat{z}_0$ called the acceleration and the bias correction. The $1-\alpha$ interval is given by $\left(\hat{G}^{-1}(\alpha_1),\ \hat{G}^{-1}(\alpha_2)\right)$ where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1-\hat{a}\left(\hat{z}_0 + z_{\alpha/2}\right)}\right),$$

$$\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\alpha/2}}{1-\hat{a}\left(\hat{z}_0 + z_{1-\alpha/2}\right)}\right),$$

$\Phi(.)$ is the standard normal cumulative distribution function, z_α is the α quantile of the standard normal distribution. The values of $\hat{a}$ and $\hat{z}_0$ are calculated as follows;

$$\hat{a} = \frac{\sum_{i=1}^{n}\left(\hat{R}(.) - \hat{R}(i)\right)^3}{6\left\{\sum_{i=1}^{n}\left(\hat{R}(.) - \hat{R}(i)\right)^2\right\}^{3/2}}$$

where $\hat{R}(i)$ is the maximum likelihood estimator of R using the original data excluding the i-th observation and

$$\hat{R}(.) = \frac{\sum_{i=1}^{n}\hat{R}(i)}{n}.$$

The value of $\hat{z}_0$ is given by

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\left\{\hat{R}^* < \hat{R}\right\}}{B}\right).$$

Small Sample Performance of the Intervals

For the confidence intervals with nominal confidence coefficient $(1-\alpha)$, we use the criterion of attainment of lower and upper error probabilities which are both equal to $\frac{\alpha}{2}$. Attainment of lower and upper nominal error probabilities is important because otherwise we will use an interval with unknown error probabilities and our conclusions therefore are imprecise and can be misleading. Attainment of nominal error probabilities (assumed equal) means that if the interval fails to contain the true value of the parameter, it is equally likely to be above as to be below the true value. Users of two sided confidence intervals expect the lower and upper error probabilities to be symmetric because they are using symmetric percentiles of the approximating distributions to form their confidence intervals. However, symmetry of error probabilities may not occur due to the skewness of the actual sampling distribution Jennings (1987).

Another criterion for comparing confidence intervals is their expected lengths, obviously the shortest confidence interval among intervals having the same confidence level is the best. We have simulated the expected lengths of the three considered intervals.

A simulation study is conducted to investigate the performance of the intervals. The indices of our simulations are:

$(n_1, n_2) = (10,10), (20,20), (30,30), (40,40), (10,40), (40,10), (20,40), (40,20)$

R : The true value of R=p(X<Y) and is taken to be 0.5, 0.7, 0.9, 0.95.

For each combination of n_1, n_2 and R , 2000 samples were generated for X taking θ = 1, $\mu = 0$, and 2000 samples for Y with $\lambda = \frac{1}{R} - 1$, $\mu = 0$. The intervals are calculated, we used B = 1000 for bootstrap calculations.

The following quantities are simulated for each interval using the results of the 2000 samples; the expected width of the interval (W): The average of the widths of the 2000 intervals. Lower error rates (L): The fraction of intervals that fall entirely above the true parameter. Upper error rates (U): The fraction of intervals that fall entirely below the true parameter. Total error rates (T): The fraction of intervals that did not contain the true parameter value.

Table 1: Simulated error rates and expected lengths of the intervals

(n_1, n_2)	R		AN	TRAN	BH	LR	BTST	TRBTST	PRC	BCa
(10, 10)	0.50	L	0.0425	0.0500	0.0245	0.0275	0.0070	0.0110	0.0310	0.0160
		U	0.0455	0.0255	0.0305	0.0340	0.0110	0.0030	0.0285	0.0235
		T	0.0880	0.0755	0.0550	0.0615	0.0180	0.0140	0.0595	0.0395
		W	0.4160	0.4160	0.4230	0.3870	0.5240	0.4940	0.4210	0.4270
	0.70	L	0.0175	0.0395	0.0245	0.0210	0.0355	0.0095	0.0115	0.0235
		U	0.0475	0.0550	0.0425	0.0200	0.0650	0.0095	0.0100	0.0200
		T	0.0650	0.0945	0.0670	0.0410	0.1010	0.0190	0.0215	0.0435
		W	0.3610	0.3570	0.3230	0.4270	0.6000	0.4500	0.4470	0.3760
	0.90	L	0.0010	0.0095	0.0075	0.0180	0.0035	0.0080	0.0550	0.0255
		U	0.1080	0.0975	0.0565	0.0425	0.0195	0.0145	0.0095	0.0240
		T	0.1090	0.1070	0.0640	0.0605	0.0230	0.0225	0.0645	0.0495
		W	0.1550	0.1560	0.1630	0.1060	0.2030	0.2170	0.1640	0.1890
	0.95	L	0.0000	0.0030	0.0120	0.0175	0.0145	0.0095	0.0655	0.0230
		U	0.1370	0.1110	0.0655	0.0480	0.0270	0.0230	0.0150	0.0185
		T	0.1370	0.1140	0.0775	0.0655	0.0415	0.0325	0.0805	0.0415
		W	0.0813	0.0825	0.0863	0.0772	0.1080	0.1160	0.0872	0.1080
(20, 20)	0.50	L	0.0390	0.0500	0.0250	0.0290	0.0140	0.0145	0.0340	0.0290
		U	0.0450	0.0295	0.0305	0.0340	0.0175	0.0215	0.0325	0.0195
		T	0.0840	0.0795	0.0555	0.0630	0.0315	0.0360	0.0665	0.0485
		W	0.3018	0.3010	0.3028	0.3355	0.3354	0.3250	0.3028	0.3050
	0.70	L	0.0175	0.0275	0.0200	0.0225	0.0145	0.0125	0.0385	0.0170
		U	0.0605	0.0420	0.0430	0.0365	0.0205	0.0160	0.0195	0.0235
		T	0.0780	0.0695	0.0630	0.0590	0.0350	0.0285	0.0580	0.0405
		W	0.2546	0.2560	0.2594	0.2305	0.2835	0.2830	0.2570	0.2630
	0.90	L	0.0030	0.0115	0.0110	0.0155	0.0130	0.0170	0.0485	0.0195
		U	0.0800	0.0605	0.0455	0.0430	0.0325	0.0160	0.0135	0.0275
		T	0.0830	0.0720	0.0565	0.0585	0.0455	0.0330	0.0620	0.0470
		W	0.1103	0.1110	0.1135	0.0845	0.1249	0.1300	0.1134	0.1230
	0.95	L	0.0035	0.0060	0.0125	0.0190	0.0200	0.0235	0.0490	0.0270
		U	0.0850	0.0855	0.0470	0.0380	0.0225	0.0240	0.0125	0.0260
		T	0.0885	0.0915	0.0595	0.0570	0.0425	0.0475	0.0615	0.0530
		W	0.0585	0.0586	0.0610	0.0558	0.0665	0.0682	0.0604	0.0662
(30, 30)	0.50	L	0.0305	0.0455	0.0255	0.0265	0.0175	0.0205	0.0290	0.0220

(n_1, n_2)	R		AN	TRAN	BH	LR	BTST	TRBTST	PRC	BCa
		U	0.0310	0.0265	0.0265	0.0270	0.0190	0.0210	0.0280	0.0255
		T	0.0615	0.0720	0.0520	0.0535	0.0365	0.0415	0.0570	0.0475
		W	0.2488	0.2480	0.2461	0.3249	0.2663	0.2600	0.2492	0.2500
	0.70	L	0.0205	0.0320	0.0225	0.0225	0.0180	0.0210	0.0400	0.0230
		U	0.0565	0.0435	0.0345	0.0355	0.0240	0.0230	0.0230	0.0255
		T	0.0770	0.0755	0.0570	0.0580	0.0420	0.0440	0.0630	0.0485
		W	0.2097	0.2100	0.2129	0.2060	0.2249	0.2240	0.2110	0.2140
	0.90	L	0.0035	0.0100	0.0090	0.0155	0.0155	0.0180	0.0365	0.0320
		U	0.0600	0.0610	0.0395	0.0305	0.0205	0.0220	0.0125	0.0255
		T	0.0635	0.0710	0.0485	0.0460	0.0360	0.0400	0.0490	0.0575
		W	0.0903	0.0907	0.0922	0.0762	0.0977	0.0999	0.0919	0.0968
	0.95	L	0.0030	0.0080	0.0145	0.0210	0.0225	0.0225	0.0425	0.0235
		U	0.0700	0.0645	0.0470	0.0320	0.0270	0.0250	0.0175	0.0275
		T	0.0730	0.0725	0.0615	0.0530	0.0495	0.0475	0.0600	0.0510
		W	0.0479	0.0480	0.0480	0.0449	0.0520	0.0529	0.0489	0.0526
(40, 40)	0.50	L	0.0300	0.0380	0.0295	0.0260	0.0180	0.0210	0.0255	0.0240
		U	0.0335	0.0185	0.0335	0.0290	0.0230	0.0155	0.0280	0.0205
		T	0.0635	0.0565	0.0630	0.0550	0.0410	0.0365	0.0535	0.0445
		W	0.2163	0.2160	0.2164	0.2989	0.2271	0.2240	0.2162	0.2170
	0.70	L	0.0170	0.0320	0.0165	0.0255	0.0210	0.0270	0.0350	0.0220
		U	0.0470	0.0280	0.0345	0.0295	0.0235	0.0170	0.0245	0.0260
		T	0.0640	0.0600	0.0510	0.0550	0.0445	0.0440	0.0595	0.0480
		W	0.1811	0.1830	0.1819	0.2003	0.1906	0.1920	0.1819	0.1850
	0.90	L	0.0090	0.0165	0.0170	0.0210	0.0220	0.0225	0.0395	0.0225
		U	0.0605	0.0470	0.0405	0.0350	0.0230	0.0235	0.0190	0.0270
		T	0.0695	0.0635	0.0575	0.0560	0.0450	0.0460	0.0585	0.0495
		W	0.0782	0.0791	0.0796	0.0687	0.0829	0.0848	0.0792	0.0824
	0.95	L	0.0050	0.0035	0.0145	0.0215	0.0265	0.0160	0.0325	0.0265
		U	0.0560	0.0575	0.0300	0.0390	0.0190	0.0245	0.0165	0.0255
		T	0.0610	0.0610	0.0445	0.0605	0.0455	0.0405	0.0490	0.0520
		W	0.0415	0.0414	0.0425	0.0400	0.0441	0.0443	0.0421	0.0442
(10, 40)	0.50	L	0.0270	0.0345	0.0415	0.0250	0.0175	0.0205	0.0530	0.0485
		U	0.0490	0.0335	0.0165	0.0295	0.0100	0.0105	0.0080	0.0055
		T	0.0760	0.0680	0.0580	0.0545	0.0275	0.0310	0.0610	0.0540
		W	0.3359	0.3350	0.3345	0.3369	0.3828	0.3700	0.3351	0.3370
	0.70	L	0.0105	0.0215	0.0375	0.0205	0.0115	0.0185	0.0790	0.0435
		U	0.0855	0.0585	0.0230	0.0395	0.0155	0.0080	0.0065	0.0085
		T	0.0960	0.0800	0.0605	0.0600	0.0270	0.0265	0.0855	0.0520
		W	0.2790	0.2820	0.3033	0.2630	0.3358	0.3400	0.2770	0.2810
	0.90	L	0.0020	0.0055	0.0220	0.0175	0.0145	0.0130	0.1055	0.0625
		U	0.1185	0.0945	0.0520	0.0550	0.0195	0.0160	0.0025	0.0045
		T	0.1205	0.1000	0.0740	0.0725	0.0340	0.0290	0.1080	0.0670
		W	0.1190	0.1190	0.1440	0.0913	0.1557	0.1640	0.1181	0.1250
	0.95	L	0.0010	0.0015	0.0125	0.0175	0.0190	0.0230	0.1120	0.0700
		U	0.1265	0.1330	0.0475	0.0535	0.0170	0.0225	0.0015	0.0065
		T	0.1275	0.1340	0.0600	0.0710	0.0360	0.0455	0.1135	0.0765
		W	0.0625	0.0615	0.0720	0.0631	0.0843	0.0864	0.0614	0.0686

(20, 40)	0.50	L	0.0320	0.0260	0.0320	0.0280	0.0210	0.0110	0.0370	0.0370
		U	0.0395	0.0290	0.0215	0.0300	0.0170	0.0205	0.0175	0.0115
		T	0.0715	0.0550	0.0535	0.0580	0.0380	0.0315	0.0545	0.0485
		W	0.2632	0.2630	0.2666	0.3312	0.2838	0.2780	0.2631	0.2650
	0.70	L	0.0175	0.0285	0.0260	0.0240	0.0185	0.0255	0.0470	0.0310
		U	0.0620	0.0415	0.0235	0.0325	0.0180	0.0160	0.0155	0.0125
		T	0.0795	0.0700	0.0495	0.0565	0.0365	0.0415	0.0625	0.0435
		W	0.2214	0.2220	0.2304	0.2086	0.2434	0.2430	0.2216	0.2240
	0.90	L	0.0025	0.0055	0.0195	0.0170	0.0180	0.0110	0.0625	0.0340
		U	0.0830	0.0840	0.0390	0.0360	0.0230	0.0215	0.0075	0.0180
		T	0.0855	0.0895	0.0585	0.0530	0.0410	0.0325	0.0700	0.0520
		W	0.0950	0.0942	0.0987	0.0797	0.1077	0.1090	0.0953	0.1010
	0.95	L	0.0040	0.0025	0.0135	0.0185	0.0160	0.0190	0.0715	0.0410
		U	0.0940	0.0825	0.0420	0.0410	0.0240	0.0245	0.0090	0.0180
		T	0.0980	0.0850	0.0555	0.0595	0.0400	0.0435	0.0805	0.0590
		W	0.0498	0.0494	0.0524	0.0462	0.0571	0.0576	0.0499	0.0540
(40, 20)	0.50	L	0.0430	0.0500	0.0230	0.0325	0.0170	0.0160	0.0220	0.0210
		U	0.0315	0.0170	0.0315	0.0310	0.0230	0.0200	0.0470	0.0230
		T	0.0745	0.0670	0.0545	0.0635	0.0400	0.0360	0.0690	0.0440
		W	0.2631	0.2630	0.2666	0.3289	0.2839	0.2770	0.2631	0.2640
	0.70	L	0.0205	0.0360	0.0145	0.0245	0.0135	0.0170	0.0235	0.0195
		U	0.0465	0.0340	0.0475	0.0265	0.0225	0.0235	0.0305	0.0260
		T	0.0670	0.0700	0.0620	0.0510	0.0360	0.0405	0.0540	0.0455
		W	0.2227	0.2240	0.2171	0.2084	0.2373	0.2370	0.2258	0.2260
	0.90	L	0.0070	0.0135	0.0140	0.0230	0.0190	0.0175	0.0275	0.0180
		U	0.0550	0.0500	0.0470	0.0305	0.0240	0.0235	0.0290	0.0330
		T	0.0620	0.0635	0.0610	0.0535	0.0430	0.0410	0.0565	0.0510
		W	0.0973	0.0982	0.0944	0.0795	0.1031	0.1060	0.1015	0.1040
	0.95	L	0.0045	0.0080	0.0125	0.0265	0.0145	0.0180	0.0195	0.0235
		U	0.0550	0.0510	0.0400	0.0300	0.0205	0.0220	0.0230	0.0250
		T	0.0595	0.0590	0.0525	0.0565	0.0350	0.0400	0.0425	0.0485
		W	0.0518	0.0521	0.0506	0.0467	0.0548	0.0560	0.0545	0.0558
(40, 10)	0.50	L	0.0525	0.0605	0.0185	0.0325	0.0090	0.0145	0.0120	0.0160
		U	0.0260	0.0140	0.0400	0.0255	0.0210	0.0175	0.0625	0.0370
		T	0.0785	0.0745	0.0585	0.0580	0.0300	0.0320	0.0745	0.0530
		W	0.3354	0.3350	0.3345	0.3402	0.3839	0.3670	0.3349	0.3330
	0.70	L	0.0370	0.0425	0.0175	0.0335	0.0120	0.0120	0.0080	0.0160
		U	0.0380	0.0295	0.0425	0.0245	0.0240	0.0235	0.0610	0.0340
		T	0.0750	0.0720	0.0600	0.0580	0.0360	0.0355	0.0690	0.0500
		W	0.2900	0.2890	0.2774	0.2740	0.3194	0.3130	0.2993	0.2890
	0.90	L	0.0070	0.0275	0.0105	0.0240	0.0100	0.0185	0.0110	0.0205
		U	0.0565	0.0520	0.0505	0.0285	0.0260	0.0255	0.0460	0.0325
		T	0.0635	0.0795	0.0610	0.0525	0.0360	0.0440	0.0570	0.0530
		W	0.1293	0.1320	0.1277	0.0901	0.1377	0.1440	0.1430	0.1370
	0.95	L	0.0055	0.0140	0.0065	0.0240	0.0170	0.0185	0.0125	0.0190
		U	0.0600	0.0585	0.0495	0.0325	0.0275	0.0285	0.0435	0.0345
		T	0.0655	0.0725	0.0560	0.0565	0.0445	0.0470	0.0560	0.0535
		W	0.0703	0.0706	0.0698	0.0672	0.0745	0.0768	0.0798	0.0755

Conclusion

Our simulations indicate that the performance of intervals based on asymptotic normality (AN intervals) are not satisfactory even for relatively large samples, they are quite anti-conservative in the sense that their coverage probabilities are often higher than the nominal confidence level. Also they are quite asymmetric, especially for values of *R* far from 0.5. The performance of the intervals based on the transformed maximum likelihood estimator (TRAN intervals) is about similar to that of AN intervals, but their anti-conservativeness and asymmetry being slightly less severe than AN intervals. Concerning Bai and Hong (BH) intervals, they often attain the nominal sizes but are asymmetric for values of *R* away from 0.5. On the other hand, the Likelihood ratio (LR) intervals attain the nominal size and are almost symmetric even for small sample sizes.

For the Bootstrap intervals, it appears that the bootstrap – t intervals (BTST) and (TRBTST) are symmetric but tend to be conservative for small sample sizes, while the percentile interval (PRC) attains the nominal level but tends to be asymmetric for values of *R* far from 0.5. The bias corrected and accelerated interval appear to be the best interval based on the bootstrap principle, they attain the nominal level and are symmetric in almost all situations considered.

With regard to interval widths, our simulation results suggest that all intervals have about equal performance. No intervals appear to be uniformly shorter or longer than the others.

Overall, the (**BCa**) interval appears to have the best performance according to the criteria of attainment of coverage probability, symmetry and expected length followed by the (LR) intervals. Although the other intervals (especially AN intervals) are anti-conservative and sometimes extremely asymmetric, which limit their usefulness, especially when lower or upper confidence bounds are desired.

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